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AN OPERATOR INEQUALITY IMPLYING CHAOTIC ORDER

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ABSTRACT

This paper proves the assertion that if positive invertible operators A and B satisfy an operator inequality $\left(B^{\frac{t}{2}}A^{\frac{S-t}{2}}B^{S-t}A^{\frac{S-t}{2}}B^{\frac{t}{2}}\right)^{\frac{1}{2s-t}} \geq$ B for0 $<$ t $< \frac{S}{2}$, then by A ≥ B, if s< 2 – t.If s≥ 2+t is additionally assumed then A ≥ B. A preliminary result Theorem 2 of J.J Fuji, M. Fuji and R. Nakamoto (FFN)[1] is further generalized in Theorem 3.

KEYWORDS: Operator Monotone Function, Operator Inequality, Chaotic Order, Hadamard-Schur Product

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1. INTRODUCTION

We use a capital letter to denote an operator. An operator means a bounded linear operator acting on a Hilbert space H. An operator A on H is said to positive in (in symbol: A≥ 0) if (Ax, x)≥ 0 for all x ∈ H and strictly positive (in symbol: A>0) if A is positive and invertible. The usual order A ≥ B for self-adjoint operators A and B on H is defined as (Ax, x) ≥ (Bx, x) for all x ∈ H.

The well known Lowner – Heinz inequality finds its extension in some other well-known inequalities like Furuta inequalities [2] for usual order, and its further extension under the Chaotic order first discussed by T. Ando [3] and then by M. Fuji [4]. Furuta [2] relaxed the restriction p∈ $[0, 1]$ as an extension of (LH) which is as follows:

Theorem (LH): Lowner – Heinz inequality

$$A \geq B \Rightarrow A^p \geq B^p$$

If and only if p ∈ [0, 1].

Theorem F. Furuta inequality:

A ≥ B ≥ 0 assures

$$A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}} \tag{1.2}$$

and

$$(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}} \quad (1.3)$$

hold for $r \geq 0, p \geq 0$ and $q \geq 1$ with

$$(1+2r)q \geq p+2r \quad (1.4)$$

The following theorem by M. Fujii, T. Furuta and E. Kamei (FFK) [5] is equivalent to Furuta inequality from the viewpoint of Kamei's satellite theorem [6] and Uchiyama's work [7].

Theorem FFK. The following statement (1)–(3) are mutually equivalent for A, B > 0

- $\log A \geq \log B$
- $A^p \geq \left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{2}}$ for $p \geq 0$
- $A^r \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)$ for $p, r \geq 0$.

We consider the following operator inequality:

$$\left(B^{\frac{s}{2}} A^{\frac{s-t}{2}} B^t A^{\frac{s-t}{2}} B^{\frac{s}{2}}\right)^{\frac{1}{2s}} \geq B \quad (1.5)$$

for the operators A, B ≥ 0 As an application of Daleckii-Krein formula (see [8]) for the derivative of matrix-valued function, one of the authors in [5] proved that if matrices A, B satisfy (1.5) for t > 1 and s =1, then $\log B \geq \log A$.

FFN[1] proved further that the usual order B $\geq$ A follows under certain restrictions on s and t. The theorem is as follows:

Theorem FFN[1]. If A, B> 0 satisfy the inequality (1.5) i.e.

$$\left(B^{\frac{s}{2}} A^{\frac{s-t}{2}} B^t A^{\frac{s-t}{2}} B^{\frac{s}{2}}\right)^{\frac{1}{2s}} \geq B$$

for some t> s > 0. Then the following assertions hold:

- If $t \geq 3s-2 \geq 0$ then $\log B \geq \log A$, and if the additional condition $t \geq s+2$ is assumed, then $B \geq A$.
- If $0 < s < \frac{1}{2}$, then $\log B \geq \log A$ *and if the additional condition* $t \geq s+2$ is assumed then $B \geq A$.

2. A PRELIMINARY RESULT FOR THE CHAOTIC ORDER

As a preliminary result the authors FFN[1] generalized a result announced by one of the authors in [5] which is as follows:

Theorem1.If positive definite matrices A, B > 0 satisfy

$$\left(B^{\frac{1}{2}} A^{\frac{1-t}{2}} B^t A^{\frac{1-t}{2}} B^{\frac{1}{2}}\right)^{\frac{1}{2}} \geq B$$

for all t > 1, then $\log B \geq \log A$.

The generalized result of FFN[1] is as follows:

Theorem2. For positive definite matrics A, B > 0, if there exists α, β such that $\alpha + \beta = 1$ and

$$\left(B^{\frac{\alpha+\beta t}{2}} A^{\frac{1-t}{2}} B^{\beta+\alpha t} A^{\frac{1-t}{2}} B^{\frac{\alpha+\beta t}{2}}\right)^{\frac{1}{2}} \geq B$$

for all t > 1, then log B ≥ log A.

We further generalize Theorem 1 using the techniques of FFN[1] as follows

Theorem 3. If the positive definite matrics A, B > 0 satisfy

$$\left(B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}}\right)^{\frac{1}{2}} \geq B$$

for $\alpha + \beta = 1$ and for all t > 1, then logB≥logA.

Proof Let $f(x) = x^{\frac{1}{2}}$, F(t) = $B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}}$ and U_t be matrices such that $U_t^* F(t)\ U_t = D(t) = diag\left(d_1(t), ..., d_n(t)\right)$, diagonal matrices. We recall the Daleckii – Krein formula for the derivative of matrix-valued function which is as follows:

$$\frac{df\left(F(t)\right)}{dt} = U_t\left(\left(f^{[1]}\left(d_i(t), d_j(t)\right) 0 U_t^* F(t) U_t\right)\right) U_t^*$$

Where 0 stands for the Hadamard-Schur product and $f^{[1]}(x, y)$ is the divided difference

$$f^{[1]}(x,y) = \begin{cases} \dfrac{f(x) - f(y)}{x - y}, & \text{if } x \neq y \\ f'(x), & \text{if } x = y \end{cases}$$

Let B itself be a diagonal matrix diag(d_j), so U_1=I, the identity matrix. Hence at t=1, we have

$$\frac{df(F)}{dt}(1) = f^{[1]}\left(d_i^2, d_j^2\right) 0 \dot{F}(1)$$

And $\left(f^{[1]}\left(d_i^2, d_j^2\right)\right) = \left(\dfrac{d_i - d_j}{d_i^2 - d_j^2}\right) = \left(\dfrac{1}{d_i + d_j}\right)$.

It follows that

$$\dot{F}(t) = B^{\frac{-t\alpha+(1+2t)\beta}{2}} (\log B)\left(\frac{-\alpha + 2\beta}{2}\right) A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}}$$

$$+ B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} (\log B) B^{-\beta+2\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}}$$

$$+ B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}} (\log B)\left(\frac{-\alpha+2\beta}{2}\right)$$

$$- \quad \frac{1}{2} B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} (\log A) B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha+(1+2t)\beta}{2}}$$

$$- \quad \frac{1}{2} B^{\frac{-t\alpha+(1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta+(1+2t)\alpha} A^{\frac{1-t}{2}} (\log A) B^{\frac{-t\alpha+(1+2t)\beta}{2}}$$

$$\rightarrow F'(1) = (\log B)\frac{(2\beta-\alpha)}{2} B^{\frac{(-\alpha+3\beta)}{2}+(-\beta+3\alpha)+\frac{(-\alpha+3\beta)}{2}}$$

$$+(\log B)(2\alpha-\beta) B^{\frac{-\alpha+3\beta}{2}+(-\beta+3\alpha)+\frac{-\alpha+3\beta}{2}}$$

$$+(\log B)\frac{(2\beta-\alpha)}{2} B^{\frac{(-\alpha+3\beta)}{2}+(-\beta+3\alpha)+\frac{(-\alpha+3\beta)}{2}}$$

$$-\frac{1}{2}.\log A.B^{\frac{(-\alpha+3\beta)}{2}+(-\beta+3\alpha)+\frac{(-\alpha+3\beta)}{2}}$$

$$-\frac{1}{2}\log A.B^{\frac{(-\alpha+3\beta)}{2}+(-\beta+3\alpha)+\frac{(-\alpha+3\beta)}{2}}$$

$$= \log B\left\{\frac{(2\beta-\alpha)}{2} + (2\alpha-\beta) + \frac{(2\beta-\alpha)}{2}\right\} B^2 - \frac{1}{2} B^{\frac{1}{2}} (\log A) B^{\frac{3}{2}} - \frac{1}{2} B^{\frac{3}{2}} (\log A) B^{\frac{1}{2}}$$

$$= \frac{1}{2} B^{\frac{1}{2}} \left(B(\log B - \log A) + (logB - \log A)B\right) B^{\frac{1}{2}}$$

$$= \frac{1}{2}(L_B + R_B)\left(B^{\frac{1}{2}} (logB - \log A) B^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left((d_i + d_j)\right)0\left(B^{\frac{1}{2}} (logB - \log A) B^{\frac{1}{2}}\right)$$

as t $\rightarrow$1, so we have

$$\frac{df(F)}{dt}(1) = \left(f^{[1]}\left(d_i, d_j\right) 0 \dot{F}(1) \right)$$

$$= \left(\frac{1}{d_i + d_j} \right) 0 \left(\frac{1}{2}\left(\left(d_i + d_j\right)\right) 0 \left(B^{\frac{1}{2}} \left(logB - \log A \right) B^{\frac{1}{2}} \right) \right)$$

$$= \frac{1}{2}\left(B^{\frac{1}{2}} \left(logB - \log A \right) B^{\frac{1}{2}} \right).$$

On the other hand, since

$$\frac{df(F)}{dt}(1) = \lim_{t \to 1} \frac{B^{\frac{-t\alpha + (1+2t)\beta}{2}} A^{\frac{1-t}{2}} B^{-t\beta + (1+2t)\alpha} A^{\frac{1-t}{2}} B^{\frac{-t\alpha + (1+2t)\beta}{2}} - B}{t - 1}$$

Weobtain $B^{\frac{1}{2}} \left(logB - \log A \right) B^{\frac{1}{2}} \geq 0$, i.e., logB ≥logA

3. MAINRESULT

The operator inequality (1.5) is the generalized form of the Furuta inequality. Taking Furuta inequality at the base we generalize Theorem 3 for the two variables in the following theorem.

Theorem4. Let A, B > 0 satisfy the inequality

$$\left(B^{\frac{t}{2}} A^{\frac{s-t}{2}} B^{s-t} A^{\frac{s-t}{2}} B^{\frac{t}{2}} \right)^{\frac{1}{2s-t}} \geq B \tag{1}$$

for $0 < t < \frac{s}{2}$, Then logA ≥ logB if s< 2 – t if an additional condition s≥2 + t is assumed then A ≥ B.

Proof. By the Furuta inequality, We have, A ≥B≥0 assures

$$\left(B^r A^p B^r \right)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}} \tag{2}$$

for r ≥ 0, p ≥ 0 and q ≥ 1 with (1+2r)q ≥ p+2r.

Since A,B>0 satisfy inequality (1), we have from Furuta inequality (2) for p = 2s-t, 2r = s - 2t

$$\left(B^{\frac{s-2t}{2}} B^{\frac{t}{2}} A^{\frac{s-t}{2}} B^{s-t} A^{\frac{s-t}{2}} B^{\frac{t}{2}} B^{\frac{s-2t}{2}} \right)^{\frac{s-2t+1}{3(s-t)}} \geq B^{s-2t+1}$$

as $\frac{1}{q} \le \frac{l+2r}{p+2r} = \frac{2s-t+1}{3(s-t)}$ and $\frac{p+2r}{q} \le l+2r = 2s-t+1$

i.e. $\left(B^{\frac{s-t}{2}} A^{\frac{s-t}{2}} B^{s-t} A^{\frac{s-t}{2}} B^{\frac{s-t}{2}} \right)^{\frac{s-2t+1}{3(s-t)}} \ge B^{s-2t+1}$

or $\left(B^{\frac{s-t}{2}} A^{\frac{s-t}{2}} B^{\frac{s-t}{2}} \right)^{\frac{2(s-2t+1)}{3(s-t)}} \ge B^{s-2t+1}$

as s<2-t, we have s-3t<2-4t and consequently 3(s-t) < 2(s-2t+1) i.e. $\frac{2(s-2t+1)}{3(s-t)}$ >1.

So we have

$$B^{\frac{s-t}{2}} A^{\frac{s-t}{2}} B^{\frac{s-t}{2}} \ge B^{\frac{3(s-t)}{2}}$$

i.e. $A^{\frac{s-t}{2}} \ge B^{\frac{s-t}{2}}$.

It implies that logA $\ge$logB by s > t and operator monotonicity of logarithmic function. Moreover if s $\ge$ 2+t then $\frac{s-t}{2} \ge 1$ and so A$\ge$B by Lowner-Heinz theorem.

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